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AN ACTIVATION CRITERION FOR REPEATED USE OF AN OPTIMAL FIXED TIME CONSTANT ENERGY REGULATOR

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SUMMARY: A regulator for linear plants is proposed whose mode of operation allows plant perturbations to go unchecked as long as they are within some bounding hypersurface in perturbed state space. This bounding hypersurface represents some limit on acceptable plant perturbations. When a perturbed state attains the hypersurface a linear feedback control is initiated, optimally driving the state back toward the origin for a fixed amount of time with a fixed amount of control energy. At control shutdown there exists some residual perturbation and the mode of operation continues.

A conventional solution for the optimal control for each operation leads to an equation for an undetermined multiplier which depends on the initial phase of the plant. With the above mode of operation the initial phase is not known a priori so that an on-line solution to the multiplier equation is necessary to specify the control prior to each operation. It is shown in this paper how this requirement may be avoided by interpreting the multiplier equation geometrically as a family of hypersurfaces in state space. By utilizing one of these hypersurfaces as the bounding hypersurface required in the mode of operation, the optimal control law may be specified before the regulator is put into service. The results of the regulator applied to a linear oscillator are given to illustrate its mode of operation.

PROBLEM STATEMENT

The system considered may be written in vector-matrix notation

$$\dot{x} = F(t)x + G(t)u \qquad \left(\cdot = \frac{d}{dt}\right) \tag{1}$$

where $x \epsilon R^n$ is the plant perturbation from equilibrium and $u \epsilon R^m$ is the control. The system is assumed to be completely controllable.[1]

The regulator's mode of operation allows plant perturbations due to disturbances and possibly plant instability with no control action as long as they are within some bounding hypersurface in perturbed state space. This bounding hypersurface represents the limiting value of acceptable plant perturbations. When a perturbation attains this hypersurface, say at time t_o, the control is initiated and operates for a fixed amount of time, $t_1 - t_o$, to drive the perturbed state back toward the origin. At t_1 the control is shut down and the regulator's cycle repeats. This paper deals with the control and the bounding hypersurface which determines the initial phase (x_o, t_o) for each operation.

The control is to be designed so that for each operation it (i) minimizes the error criterion

$$J = \frac{1}{2}\left\|x(t_1)\right\|_A^2 + \frac{1}{2}\int_{t_o}^{t_1}\left\|x(t)\right\|_{Q(t)}^2 dt \tag{2}$$

where A and $Q(t)$ are finite symmetric nonnegative definite matrices at least one of which is positive definite and $Q(t) \epsilon C^2$ for all $t \epsilon \left[t_o, t_1\right]$ and (ii)

NASA TM X-57169

This paper was presented at the Twelfth East Coast Conference on Aerospace and Navigational Electronics, October 27-29, 1965, in Baltimore, Md.

izes a given control energy

$$E = \frac{1}{2}\int_{t_o}^{t_1} \left\| u(t) \right\|^2_{R(t)}\, dt \qquad (3)$$

where $R(t)$ is a symmetric positive definite matrix and $R(t)\epsilon C^2$ for all $t\epsilon\left[t_o, t_1\right]$. The design admits plant perturbations at shutdown and the control energy, E, may therefore be specified less than the minimum energy required to get to the origin. This procedure avoids an unbounded control and other associated problems. The minimum energy, E^o, is given by

$$E^o = \frac{1}{2}\left\| x_o \right\|^2_{W^{-1}(t_o,t_1)} \qquad (4)$$

where $W(t_o,t_1)$ is the controllability matrix.[1] The bounding hypersurface must, therefore, be such that (i) it lies within the hypersurface defined by the equation

$$E = \frac{1}{2}\left\| x_o \right\|^2_{W^{-1}(t_o,t_1)} \qquad (5)$$

and that (ii) all perturbations within and on the bounding hypersurface are acceptable. Requirement (ii) depends on the accuracy with which the linear equations represent the physical system as well as the designer's needs.

SOLUTION FOR THE CONTROL

The first step is to adjoin (3) to (2) with a constant positive real Lagrange multiplier to give as the adjoined criterion

$$V = \frac{1}{2}\left\| x(t_1) \right\|^2_A + \frac{1}{2}\int_{t_o}^{t_1} \left\{ \left\| x(t) \right\|^2_{Q(t)} + \lambda \left\| u(t) \right\|^2_{R(t)} \right\} dt \qquad (6)$$

As in reference 2 the conjugate variable is denoted $\xi\epsilon R^n$ and the Hamiltonian is written

$$H(x,\xi,u,t) = \frac{1}{2}\left\{ \left\| x \right\|^2_{Q(t)} + 2\xi \cdot \left[F(t)x + G(t)u\right] + \lambda \left\| u \right\|^2_{R(t)} \right\} \qquad (7)$$

The Hamiltonian is then minimized with respect to u to give as the optimal control function

$$u^o(t) = -\frac{1}{\lambda} R^{-1}(t)G'(t)\xi(t) \qquad (8)$$

where ξ is related to the system through the canonic equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} F(t) & -\frac{1}{\lambda} R^{-1}(t)G'(t) \\ -Q(t) & -F'(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} \qquad (9)$$

This set of equations have the boundary conditions[2]

$$x(t_o) = x_o \qquad (10)$$

$$\xi(t_1) = Ax(t_1) \qquad (11)$$

To solve for $\xi(t)$ let $\begin{bmatrix} X(t,t_1) \\ \Xi(t,t_1) \end{bmatrix}$ be a set of solutions to (9) such that

$$X(t_1,t_1) = I \qquad (12)$$

$$\Xi(t_1,t_1) = A \qquad (13)$$

Then by linearity

$$x(t) = X(t,t_1)x(t_1) \qquad (14)$$

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$$\xi(t) = \Xi(t,t_1)x(t_1) \qquad (15)$$

In reference 2 $X(t,t_1)$ is shown to be the fundamental matrix of the optimally controlled plant. Then with the energy level, E, chosen such that $x(t_1) \neq 0$, $X^{-1}(t,t_1)$ exists for all $t\epsilon[t_o,t_1]$ and $\xi(t)$ may be written

$$\xi(t) = \Xi(t,t_1)X^{-1}(t,t_1)x(t) \qquad (16)$$

Equations (16) and (8) give for the optimal control

$$K^o(t,x) = -\frac{1}{\lambda} R^{-1}(t)G'(t)\Xi(t,t_1)X^{-1}(t,t_1)x \qquad (17)$$

where $K^o(t,x)$ replaces $u^o(t)$ to denote optimal feedback control. λ has yet to be determined; however, we may examine its relation to the specific energy level, E, the fixed time, $t_1 - t_o$, and the system dynamics.

Substitution of (15) into (8) gives the control in terms of the terminal state. This then may be substituted into (3) to give as the constraint equation

$$E = \left\| x(t_1) \right\|^2_D \qquad (18)$$

where

$$D = \frac{1}{2\lambda^2} \int_{t_o}^{t_1} \Xi'(t,t_1)G(t)R^{-1}(t) \times G'(t)\Xi(t,t_1)dt \qquad (19)$$

Then by utilizing (14) in (18), the final form of the energy constraint equation is

$$E = \left\| x_o \right\|^2_B \qquad (20)$$

where

$$B = X^{-1'}(t_o,t_1)DX^{-1}(t_o,t_1) \qquad (21)$$

Now the standard solution to the problem would require the solution of (20) for a positive real multiplier prior to each operation since from (20) and (21) it is clear that, in general, λ is a function of the initial phase (x_o,t_o). To this end an analog method for continuously tracking the multiplier as a function of the phase for the case when $Q(t) = 0$ is essentially given in reference 3. A method will now be shown whereby the on-line tracking or computing of λ can be avoided by employing the aforementioned concept of a bounding hypersurface.

THE BOUNDING HYPERSURFACE

The regulator's mode of operation requires that some activation criterion be used to begin each control operation. In view of this, it is noted that for each t_o and with λ as a parameter, (20) defines a family of hypersurfaces in state space from which the optimal control associated with λ will use E amount of energy. If, for each t_o, there is a member of this family suitable for a bounding hypersurface, then by using these hypersurfaces to define a bounding hypersurface varying with t_o, the on-line computation of $\lambda(t_o)$ may be avoided.

The method of using this activation criterion is to feed back the state with time varying gains to form

$$e = \left\| x_o \right\|^2_{B(\lambda(t_o),t_o)} \qquad (22)$$

As long as e is less than E the control remains off, but when e equals E the control is initiated and the optimal feedback control law associated with the $\lambda(t_o)$ is used. Of course, for time constant systems there will only be one λ so the bounding hypersurface will not be time varying.

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It should be pointed out that (18) is a direct indication of the effectiveness of the system design since for each t_o it defines the hypersurface to which the control will drive the state.

A SIMPLE EXAMPLE

In the interest of simplicity, the example plant is a constant undamped harmonic oscillator. In spite of its simplicity, it illustrates the regulator's usefulness on the class of constant neutrally stable systems subject to small disturbances. The symmetric spinning space vehicle is a member of this class in which considerable interest has been shown.[4,5,6,7]

The vector-matrix equation is written

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (23)$$

where x_1 is the displacement and x_2 is the velocity. The system is required to run from $t_o = 0$ to $t_1 = T$. The controllability matrix is then computed with the result

$$W(o,T) = \frac{1}{2} \begin{bmatrix} \frac{T}{2} - \frac{1}{4} \sin 2T & \frac{1}{4} - \frac{1}{4} \cos 2T \\ \frac{1}{4} - \frac{1}{4} \cos 2T & \frac{T}{2} + \frac{1}{4} \sin 2T \end{bmatrix} \qquad (24)$$

$W(o,T)$ may be shown to be positive definite for all $T \neq \emptyset$ so the system is controllable.

To further simplify the problem, only the terminal error will be minimized and the A matrix will be the identity matrix. This makes the error criterion circular in nature

$$J = \frac{1}{2}\left[x_1^2(T) + x_2^2(T)\right] \qquad (25)$$

This norm is constant for the free plant and is a good measure of the system perturbation from equilibrium. Typically, neutrally stable systems have constants of free motion which may be expressed as quadratic forms in the state variables. One of these forms is often useful as an error criterion.

The energy constraint is chosen as half the integral square energy by letting $R = 1$ which gives

$$E = \frac{1}{2} \int_0^T u^2(t)dt \qquad (26)$$

This completes the problem statement. Use of equation (9) results in the canonic differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -\frac{1}{\lambda} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \xi_1 \\ \xi_2 \end{bmatrix} \qquad (27)$$

The solution of (27) with the appropriate terminal conditions results in

$$\Xi(t,T) = \begin{bmatrix} \cos(T - t) & -\sin(T - t) \\ \sin(T - t) & \cos(T - t) \end{bmatrix} \qquad (28)$$

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and

$$X^{-1}(t,T) = \frac{1}{\Delta}\begin{bmatrix} (2\lambda+T-t)\cos(T-t)+\frac{1}{2\lambda}\sin(T-t) & \frac{(2\lambda+T-t)}{2\lambda}\sin(T-t) \\ \frac{-(2\lambda+T-t)}{2\lambda}\sin(T-t) & \frac{(2\lambda+T-t)}{2\lambda}\cos(T-t) - \frac{1}{2\lambda}\sin(T-t) \end{bmatrix} \quad (29)$$

where

$$\Delta = \left[1 + \frac{1}{\lambda}(T - t)\right]^2 - \left[\frac{1}{2\lambda}\sin(T - t)\right]^2 \quad (30)$$

Direct substitution of (28) and (29) into (17) and (19) results in the optimal feedback control law

$$K^o(t,x) = \frac{-1}{\lambda\Delta}\left[\left(\frac{1}{2\lambda}\sin^2(T - t)\right)x_1 + \left(\frac{1}{2\lambda}(T - t) + 1 - \frac{1}{2\lambda}\sin(T - t)\cos(T - t)\right)x_2\right] \quad (31)$$

and the D matrix

$$D = \frac{1}{2\lambda^2}\begin{bmatrix} \frac{T}{2} - \frac{\sin 2T}{4} & \frac{\sin^2 T}{2} \\ \frac{\sin^2 T}{2} & \frac{T}{2} + \frac{\sin 2T}{4} \end{bmatrix} \quad (32)$$

With the substitution of (29) and (32) into (21), the B matrix is determined. E, T, and some point desired to lie on the control activation hypersurface may now be chosen so that λ may be computed and the activation hypersurface determined.

From $X^{-1}(o,T)$ and D, it is evident that with T an even multiple of π, B defines a circular hypersurface. This indicates that, in general, the time of operation may be a vital parameter in shaping the B-hypersurface. In the example, however, a more general case has been chosen by setting $T = 10$. Since the period of the oscillator's free motion is 2π, the control is therefore in operation for more than one period. A displacement of 5 and a zero velocity was chosen rather arbitrarily to be on the B-hypersurface. By using this point and (24) in equation (4), the minimum energy is found to be 2.62 units. The energy in this problem was chosen as 2.5 units to be within this limit. Equation (20) was then solved for λ. Its value was 0.115. The resulting B-hypersurface and D-hypersurface which are slightly elliptical are shown in figure 1. A typical trajectory during a control operation is shown between the two hypersurfaces. The time history of this trajectory and its associated control are shown in figure 2.

CONCLUSIONS

A special family of hypersurfaces in state space has been identified with the optimal linear regulator. It has been shown that on-line computation of the multiplier associated with the control constraint may be eliminated by choosing a member of this special family as the regulator's activation criterion. This

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results in a relatively simple optimal control system suitable for repetitive operation.

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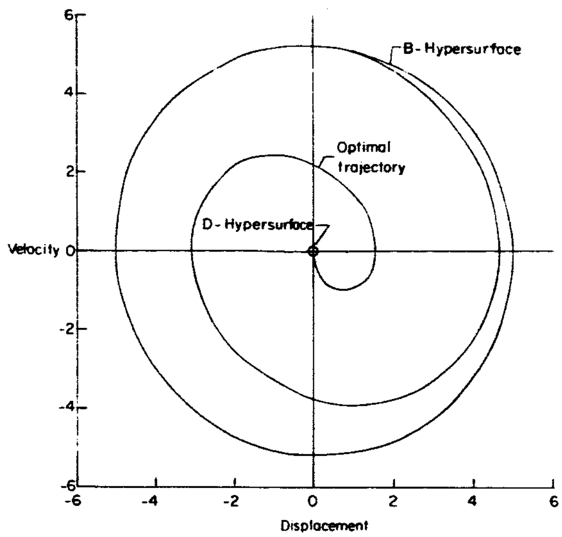

Figure 1.- Hypersurface geometry and a typical optimal trajectory.

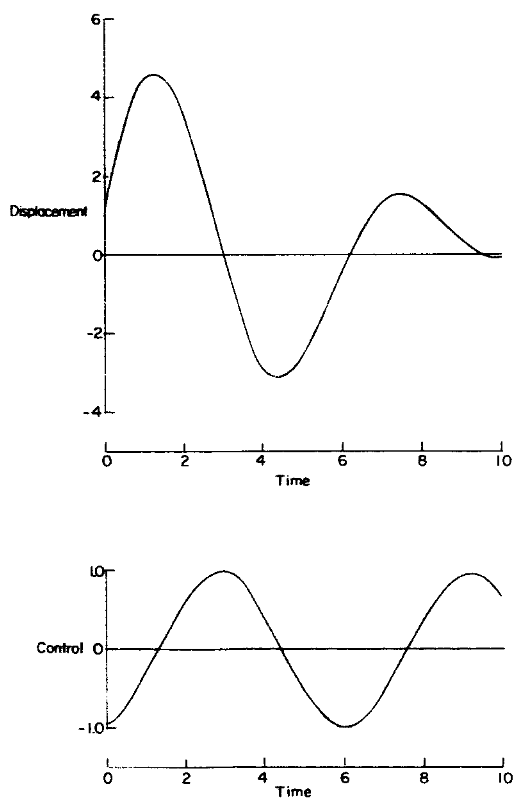

Figure 2.- Typical trajectory and control time history.

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